Homework 10 (Stats 620, Winter 2017)

Due Tuesday April 18, in class

Questions are derived from problems in Stochastic Processes by S. Ross.

1. A stochastic process \( \{X(t), t \geq 0\} \) is said to be stationary if \( X(t_1), \ldots, X(t_n) \) has the same joint distribution as \( X(t_1 + a), \ldots, X(t_n + a) \) for all \( n, a, t_1, \ldots, t_n \).

(a) Prove that a necessary and sufficient condition for a Gaussian process to be stationary is that \( \text{Cov}(X(s), X(t)) \) depends only on \( t - s \), \( s \leq t \), and \( \mathbb{E}[X(t)] = c \).

(b) Let \( \{X(t), t \geq 0\} \) be Brownian motion and define

\[
V(t) = e^{-\alpha t/2}X(\alpha e^{\alpha t}).
\]

Show that \( \{V(t), t \geq 0\} \) is a stationary Gaussian process. It is called the Ornstein-Uhlenbeck process.

Solution:

If the Gaussian process is stationary then for \( t > s \)

\[
\begin{pmatrix}
X(t) \\
X(s)
\end{pmatrix} \overset{d}{=} \begin{pmatrix}
X(t - s) \\
X(0)
\end{pmatrix}
\]

Thus \( \mathbb{E}[X(s)] = \mathbb{E}[X(0)] \) for all \( s \) and \( \text{Cov}(X(t), X(s)) = \text{Cov}(X(t - s), X(0)) \) for all \( t < s \).

Now, assume \( \mathbb{E}[X(t)] = c \) and \( \text{Cov}(X(t), X(s)) = h(t - s) \). For any \( T = (t_1, \cdots, t_k) \) define vector \( X_T \equiv (X(t_1), \cdots, X(t_k))' \). Let \( T = (t_1 - a, \cdots, t_k - a) \). If \( \{X(t)\} \) is a Gaussian process then both \( X_T \) and \( X_T' \) are multivariate normal and it suffices to show that they have the same mean and covariance. This follows directly from the fact that they have the same element-wise mean \( c \) and the equal pair-wise covariances, \( \text{Cov}(X(t_i - a), X(t_j - a)) = h(t_i - t_j) = \text{Cov}(X(t_i), X(t_j)) \).

(b) Since all finite dimensional distributions of \( \{V(t)\} \) are Normal, it is a Gaussian process. Thus from part (a) it suffices to show the following:

(a) \( \mathbb{E}[V(t)] = e^{-\alpha t/2}\mathbb{E}[X(\alpha e^{\alpha t})] = 0 \). Thus \( \mathbb{E}[V(t)] \) is constant.

(b) For \( s \leq t \),

\[
\text{Cov}(V(s), V(t)) = e^{-\alpha(t+s)/2}\text{Cov}(X(\alpha e^{\alpha s}), X(\alpha e^{\alpha t})) = e^{-\alpha(t+s)/2}\alpha e^{\alpha s} = e^{-\alpha(t-s)/2},
\]

which depends only on \( t - s \).

2. Let \( X(t) \) be standard Brownian motion. Find the distribution of:

(a) \( |X(t)| \).

(b) \( |\min_{0 \leq s \leq t} X(s)| \)

(c) \( \max_{0 \leq s \leq t} X(s) - X(t) \)
**Hint:** all three parts have the same answer.

**Solution:**

(a) Let $Y(t) = |X(t)|$. For $y \geq 0$

$$F_Y(y) = P(|X(t)| \leq y)$$

$$= P(-y \leq X(t) \leq y) = 2 \int_0^y \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx = \sqrt{\frac{2}{\pi}} \int_0^{y/\sqrt{t}} \exp\left(-\frac{u^2}{2}\right) du.$$

(b) Let $Y(t) = |\min_{0 \leq s \leq t} X(s)|$. For $y \geq 0$

$$F_Y(y) = P(Y(t) \leq y) = P(\min_{0 \leq s \leq t} X(s) \geq -y)$$

$$= P(T_{-y} \geq t) = 1 - \int_{y/\sqrt{t}}^{\infty} \sqrt{\frac{2}{\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{y/\sqrt{t}} \exp\left(-\frac{u^2}{2}\right) du.$$

(c) Let $Y = \max_{0 \leq s \leq t} X(s)$ and $X = X(t)$ then

$$F(x, y) = \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x - 2y}{\sqrt{t}}\right), y \geq x, y > 0.$$

Let $\Phi$ and $\phi$ be the distribution and density functions respectively of a standard normal random variable. Using results derived in class,

$$F(x, y) = \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x - 2y}{\sqrt{t}}\right), y \geq x, y > 0.$$

Thus,

$$\frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{2}{t} \phi^\prime\left(\frac{x - 2y}{\sqrt{t}}\right).$$

Since the Jacobian for the transformation $V = Y - X, W = Y$ is of unit modulus, the density of $(V, W)$ is given by

$$f(v, w) = \frac{2}{t} \phi^\prime\left(\frac{v - w}{\sqrt{t}}\right), v, w \geq 0. \quad (1)$$

Thus

$$P(Y - X \leq y) = \mathbb{P}(V \leq y) = \int_0^y \int_0^\infty \frac{2}{t} \phi^\prime\left(\frac{v - w}{\sqrt{t}}\right) dw dv$$

$$= \int_0^y \frac{2}{\sqrt{t}} \phi^\prime\left(\frac{v}{\sqrt{t}}\right) dv$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{y/\sqrt{t}} \exp\left(-\frac{u^2}{2}\right) du. \quad (2)$$
3. Let $M(t) = \max_{0 \leq s \leq t} X(s)$ where $X(t)$ is standard Brownian motion. Show that

$$
P(M(t) > a \mid M(t) = X(t)) = e^{-a^2/2t}, \quad a > 0.
$$

**Hint:** One approach is outlined below. There may be other ways.

(i) Differentiate the expression

$$
P(M(t) > y, B(t) < x) = \int_{2y-x}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/2t} \, du
$$
to find the joint density of $M(t)$ and $B(t)$.

(ii) Transform variables to find the joint density of $M(t)$ and $M(t) - B(t)$. This involves using the Jacobian formula (e.g. Ross, A First Course in Probability, 6th edition, Section 6.7): If $X_1$ and $X_2$ have joint density $f_{X_1,X_2}(y_1, y_2)$, then (supposing suitable regularity)

$$
f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) \left| \frac{\partial h_1}{\partial y_1} \frac{\partial h_2}{\partial y_1} \frac{\partial h_1}{\partial y_2} \frac{\partial h_2}{\partial y_2} \right|
$$

where $J$ is the matrix determinant (Jacobian) given by

$$
J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}
$$

(iii) Find the conditional density of $M(t)$ given $M(t) - B(t) = 0$.

**Solution:**

$$
V = \max_{0 \leq s \leq t} X(s) - X(t) \quad \text{and} \quad W = \max_{0 \leq s \leq t} X(s).
$$

The joint density of $(V, W)$ is given by equation (1), and the marginal density of $V$ follows from equation (2):

$$
f_V(v) = \int_0^\infty \frac{2}{t} \phi \left( \frac{v-w}{\sqrt{t}} \right) \, dw = \frac{2}{\sqrt{t}} \phi \left( \frac{v}{\sqrt{t}} \right).
$$

The conditional density of $W$ given $V = 0$ is $f(0, w)/f_V(0)$, which gives

$$
P(W \leq a | V = 0) = \int_0^a \frac{f(0, w)}{f_V(0)} \, dw = 1 - \frac{\phi(-a/\sqrt{t})}{\phi(0)}.
$$

Thus

$$
P(W > a | V = 0) = 1 - P(W \leq a | V = 0) = e^{-\frac{a^2}{2t}}.
$$

4. For a Brownian motion process with drift coefficient $\mu$, let

$$
f(x) = \mathbb{E} [\text{time to hit either } A \text{ or } -B \mid X_0 = x],
$$

where $A$ and $B$ are constants.
where $A > 0, B > 0, -B < x < A$.

(a) Derive a differential equation for $f(x)$.

(b) Solve this equation.

(c) Use a limiting random walk argument (see Problem 4.22 of Chapter 4) to verify the solution in part (b).

Solution:

(a) Note that the conditional distribution of process $\{Y(t) = X(t + h) : t \geq 0 | X(h) = x\}$ is the same as distribution of $\{X(t) : t \geq 0 | X(0) = x\}$. Thus if $T(x) =$ time to hit either $A$ or $-B$ given $X(0) = x$, then

$$T(x) = h + T(X(h)) + o(h).$$

Thus for $Y = X(h) - X(0)$,

$$f(x) = \mathbb{E}[T(x)] = h + \mathbb{E}[f(x + Y)] + o(h).$$

From the Taylor series expansion

$$f(x) = h + \mathbb{E}[f(x) + f'(x)Y + f''(x)Y^2/2 + \cdots] + o(h),$$

it follows that,

$$h + f'(x)\mu h + f''(x)(\mu^2 h^2 + h)/2 + o(h) = 0.$$ 

Dividing the equation above by $h$ on both side, we obtain

$$1 + f'(x)\mu + f''(x)(\mu^2 h + 1)/2 = o(h)/h,$$

That is, letting $h \to 0$,

$$1 + f'(x)\mu + f''(x)/2 = 0. \quad (3)$$

(b) Let $v = 1 + f'(x)\mu$. Equation (3) becomes

$$v + \frac{1}{2\mu} \frac{dv}{dx} = 0$$

Thus $v = c_1 e^{-2\mu x} + f'(x)\mu$. This gives

$$f'(x) = \frac{c_1 e^{-2\mu x} - 1}{\mu}.$$ 

Finally

$$f(x) = \frac{1}{\mu} \left(\frac{c_1 e^{-2\mu x} - 1}{-2\mu} - x\right) + c_2. \quad (4)$$

Using the boundary conditions $f(A) = 0 = f(-B)$, equation (4) gives

$$f(x) = \frac{A + B}{\mu} \left(\frac{e^{-2\mu x} - e^{-2\mu A}}{e^{-2\mu A} - e^{-2\mu B}}\right) + \frac{A - x}{\mu}. \quad (5)$$

4
(c) From the class notes for $T = T(0)$,

$$
\mathbb{E}[T] = \lim_{n \to \infty} \frac{\mathbb{E}[T(n)]}{n} \\
= \lim_{n \to \infty} \frac{1}{n} \left( A P_A - B (1 - P_A) \right) \\
= \lim_{n \to \infty} \frac{1}{n} \left( A P_A - B (1 - P_A) \right) / \mu/n.
$$

Here

$$P_A \approx \frac{1 - e^{-\theta_n B}}{e^{\theta_n A} - e^{-\theta_n B}}, \tag{6}$$

where $\theta_n$ satisfies

$$\mathbb{E}[e^{\theta_n (Y_1 + \mu / \sqrt{n}) / \sqrt{n}}] = 1.$$

Thus

$$\mathbb{E}[e^{\theta_n Y_1 / \sqrt{n}}] = e^{-\theta_n \mu / n}.$$

Since $Y_1$ is standard normal. By the moment generating function

$$e^{\theta_n^2 / (2n)} = e^{-\theta_n \mu / n},$$

it follows that $\theta_n = -2\mu$. Substituting in (6), we obtain

$$P_A \approx \frac{1 - e^{2\mu B}}{e^{-2\mu A} - e^{2\mu B}}.$$

Thus

$$\mathbb{E}[T] \approx \frac{A + B}{\mu} \left( \frac{1 - e^{2\mu B}}{e^{-2\mu A} - e^{2\mu B}} \right) - \frac{B}{\mu},$$

which is the same as $f(0)$ obtained earlier.

**Recommended reading:**
Sections 8.3, 8.4, 8.5.

**Supplementary exercises:** 8.3, 8.4, 8.6, 8.16
Optional, but recommended. Do not turn in solutions—they are in the back of the book.