1. Let $A(t)$ and $Y(t)$ denote respectively the age and excess at $t$. Find:

(a) $\mathbb{P}\{Y(t) > x | A(t) = s\}$.

(b) $\mathbb{P}\{Y(t) > x | A(t + x/2) = s\}$.

(c) $\mathbb{P}\{Y(t) > x | A(t + x) > s\}$ for a Poisson process.

(d) $\mathbb{P}\{Y(t) > x, A(t) > y\}$.

(e) If $\mu < \infty$, show that, with probability 1, $A(t)/t \to 0$ as $t \to \infty$.

Hint: For (d), use a regenerative process argument (E.g. Ross, section 3.7) to find $\lim_{t \to \infty} \mathbb{P}(Y(t) > x, A(t) > y)$. For (e), you may use without proof the following results on convergence with probability 1: (L1) $\lim_{n \to \infty} S_n/n = \mu$; (L2) $\lim_{t \to \infty} N(t) = \infty$; (L3) $\lim_{t \to \infty} N(t)/t = 1/\mu$.

Solution: (a)

$$
\mathbb{P}[Y(t) > x | A(t) = s] = \mathbb{P}[X_{N(t)+1} > s + x | S_{N(t)} = t - s] = \mathbb{P}[X_1 > s + x | X_1 > s] = \frac{\mathcal{F}(s + x)}{\mathcal{F}(s)}.
$$

Here is a more formal solution:

$$
\mathbb{P}[Y(t) > x | A(t) = s] = \mathbb{P}[S_{N(t)+1} > t + x | S_{N(t)} = t - s] = \mathbb{P}[X_{N(t)+1} > s + x | S_{N(t)} = t - s]
$$

$$
= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s | S_{n} = t - s, N(t) = n] \mathbb{P}[N(t) = n | S_{N(t)} = t - s]
$$

$$
= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s | S_{n} = t - s, X_{n+1} > s] \mathbb{P}[N(t) = n | S_{N(t)} = t - s] (\text{by independence})
$$

$$
= \mathbb{P}[X_1 > s + x, X_1 > s] = \frac{\mathcal{F}(s + x)}{\mathcal{F}(s)}
$$

(b)

$$
p := \mathbb{P}[Y(t) > x | A(t + x/2) = s]
$$

For $s \geq x/2$, an argument similar to (a) allows us to write

$$
p = \mathbb{P}[\text{no event in } (t + x/2 - s, t + x)] \text{ event at } t + x/2 - s, \text{ no events in } (t + x/2 - s, t + x/2)]
$$

$$
= \mathbb{P}[X_{N(t)+1} > s + x/2 | S_{N(t)} = t + x/2 - s, X_{N(t)+1} > s]
$$

$$
= \mathbb{P}[X_1 > s + x/2, X_1 > s] = \frac{\mathcal{F}(s + x/2)}{\mathcal{F}(s)}
$$

For $s < x/2$, $\{A(t + x/2) = s\} \Rightarrow \{Y(t) \leq s - x/2\}$. It follows that $p = 0$. 

Homework 4 (Stats 620, Winter 2017)

Due Tuesday Feb 14, in class.

Questions are derived from problems in *Stochastic Processes* by S. Ross.
\[
q = \mathbb{P}[Y(t) > x | A(t + x) > y]
\]
\[
= \mathbb{P}[\text{no event in } [t, t + x] | \text{no events in } [t + x - s, t + x]]
\]

for \(0 \leq s \leq x\), since the process is Poisson with independent increments,

\[
q = \mathbb{P}[\text{no event in } [t, t + x - s]]
\]

= \exp(-\lambda(x-s)),

where \(\lambda\) is the rate of the Poisson process. For \(s > x\), \(q = 1\).

(d)

\[
\mathbb{P}(Y(t) > x, A(t) > y) = \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y)
\]
\[
= \mathbb{P}(X_1 > t + x, t > y | S_{N(t)} = 0)\mathbb{P}(S_{N(t)} = 0)
\]
\[
+ \int_0^t \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y | S_{N(t)} = s) \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x - s)dF_{S_{N(t)}}(s)
\]
\[
= \mathbb{1}_{t>y} \mathbb{P}(X_1 > t + x | X_1 > t)\mathbb{P}(S_{N(t)} = 0)
\]
\[
+ \int_0^t \mathbb{1}_{t-s>y} \mathbb{P}(X > t + x - s | X > t - s)dF_{S_{N(t)}}(s)
\]
\[
= \mathbb{1}_{t>y} \mathbb{P}(t + x) + \int_{t-y}^{t} F(t + x - s)dm(s).
\]

Let \(P_t = \mathbb{P}[Y(t) > x, A(t) > y]\). Define a regenerative process to be “on” at \(t\) if \(S_{N(t)} < t - y\) and \(S_{N(t)+1} > t + x\). Thus, \(P_t\) is the probability that the process is “on” at time \(t\). By the regenerative process limit theorem

\[
\lim_{t \to \infty} P_t = \frac{\mathbb{E}[\text{time “on” during a cycle}]}{\mathbb{E}[\text{time of the cycle}]}
\]
\[
= \frac{\mathbb{E}[\max(X_1 - (x + y), 0)]}{\mu}
\]
\[
= \frac{1}{\mu} \int_{x+y}^{\infty} (z - x - y) dF(z)
\]

(e)

\[
\lim_{t \to \infty} \frac{A(t)}{t} = \lim_{t \to \infty} \frac{t - S_{N(t)}}{t}
\]
\[
= 1 - \lim_{t \to \infty} \frac{S_{N(t)}}{N(t)} \lim_{t \to \infty} \frac{N(t)}{t}
\]
\[
= 1 - \mu/\mu \quad \text{(by (L1), (L2) and (L3))}
\]
\[
= 0
\]

2. Consider a single-server bank in which potential customers arrive in accordance with a renewal process having interarrival distribution \(F\). However, an arrival only enters the bank if the
server is free when he or she arrives; otherwise, the individual goes elsewhere without being served. Would the number of events by time \( t \) constitute a (possibly delayed) renewal process if an event corresponds to a customer:

(a) entering the bank?
(b) leaving the bank after being served?
What if \( F \) were exponential?

Solution: Let \( X_i \) denote the length of the \( i \)-th service and let \( Y_i \) denote the time from the end of \( i \)-th service until the start of the \( i+1 \)-th service. Let \( Y_0 \) denote the time when first arrival enters the bank (and gets service). Note that \( X_i \) and \( Y_i \) may be dependent when the arrival is not a Poisson process.

(a) In this case, each cycle consists of \( Z_i = X_i + Y_i, i = 1, 2, \ldots \) and \( Z_0 = Y_0 \). Since \( X_i \) and \( Y_i \) are independent of \( X_j \) and \( Y_j \) with \( j = 1, \ldots, i-1 \), \( \{Z_i\}_{i \in \mathbb{N}} \) are i.i.d copies. We thus have a delayed renewal process.

(b) In this case, \( Z_i = Y_{i-1} + X_i \). When \( X_i \) and \( Y_i \) are dependent, \( \{Z_i\}_{i \in \mathbb{N}} \) are not i.i.d. copies. We do not have a (delayed) renewal process. One counter example can be constructed as in the sequel. Suppose the service distribution is given by

\[
Y_1 = \begin{cases} 
1 & \text{w.p. 0.5} \\
10 & \text{w.p. 0.5}
\end{cases}
\]

and the interarrival times of the customers to the bank \( Z_n \sim F \) are given by, \( Z_1 = 6 \) w.p. 1. Then, given a previous interval between departures \( S_n - S_{n-1} = 3 \), we know that the next arrival will enter the bank at time \( S_n + 4 \).

If \( F \) is exponential (a) still gives a delayed renewal process. (b) now results in a (non-delayed) renewal process, since the memoryless property implies that \( Y_i \) is independent of \( X_i, i \in \mathbb{N} \). Hence, \( \{Z_i\}_{i \in \mathbb{N}} \) are i.i.d. copies.

3. On each bet a gambler, independently of the past, either wins or loses 1 unit with respective probability \( p \) and \( 1 - p \). Suppose the gambler’s strategy is to quit playing the first time she wins \( k \) consecutive bets. At the moment she quits

(a) find her expected winnings.
(b) find the expected number of bets that she has won.

Hint: It may help you to look at Example 3.5(A) in Ross.

Solution: Let

\[
Y_n = \begin{cases} 
1 & \text{if nth game is a win} \\
0 & \text{else}
\end{cases}
\]

and

\[
X_n = \begin{cases} 
1 & \text{if nth game is a win} \\
-1 & \text{else}
\end{cases}
\]

and let \( N = \inf\{n \geq k : \sum_{m=n-k+1}^n X_m = k\} \) and \( W = \sum_{i=1}^N Y_i \), the first time \( k \) consecutive games are won. Let \( W = \sum_{i=1}^N X_i \), the gamblers total winnings. Also let \( N_W = \sum_{i=1}^N Y_i \), the number of games won.
From Ross, Example 3.5A, \( E[N] = \sum_{i=1}^{k} \frac{1}{p^i} \). Also \( N \) is a stopping time w.r.t \( X_i, i = 1, 2, \ldots \). By Wald’s equation, we have
\[
E[W] = E[N] E[X_1] = \left( \sum_{i=1}^{k} \frac{1}{p^i} \right) (2p - 1)
\]

(b) \( N \) is also a stopping time w.r.t. \( Y_i, i = 1, \ldots \), so Wald’s equation gives
\[
E[NW] = E[N] E[Y_1] = \left( \sum_{i=1}^{k} \frac{1}{p^i} \right) p = \sum_{i=0}^{k-1} \frac{1}{p^i}
\]

4. Prove Blackwell’s theorem for renewal reward processes. That is, assuming that the cycle distribution is not lattice, show that, as \( t \to \infty \),
\[
E[\text{reward in} \ (t, t+a) ] \to a \frac{E[\text{reward incycle}]}{E[\text{time of cycle}]}.
\]
Assume that any relevant function is directly Riemann integrable.

Hint: You may adopt an informal approach by assuming that one can write
\[
E[\int_t^{t+a} dR(s)] = \int_t^{t+a} E[dR(s)],
\]
and then developing the identity
\[
E[dR(t)] = E[R_1|X_1 = t] dF(t) + \int_0^t \{E[R_1|X_1 = t - x] dF(t - x)\} dm(x).
\]
If you can find a more elegant or more rigorous solution, that would also be good!

Solution: Let \( R(t) \) be the reward accumulated by time \( t \). Then,
\[
E[\text{reward in} \ (t, t+a)] = E[R(t+a) - R(t)]
= E[\int_t^{t+a} dR(s)]
= \int_t^{t+a} E[dR(s)]
\]
assuming that the interchange is allowed, e.g. if \( R(t) \) is increasing. Now,
\[
E[dR(t)] = E[E[dR(t)|S_{N(t)}]]
= E[dR(t)|S_{N(t)} = 0] \mathbb{P}[S_{N(t)} = 0] + \int_0^\infty E[dR(t)|S_{N(t)} = y] dF_{S_{N(t)}}(y)
= E[dR(t)|S_{N(t)} = 0] \mathcal{F}(t) + \int_0^\infty E[dR(t)|S_{N(t)} = y] \mathcal{F}(t - y) dm(y).
\]
Now, since $R(t)$ only increases when an event occurs,

$$
\mathbb{E}[dR(t) | S_N(t) = y] = \mathbb{E}[R_{N(t)} + 1 | X_{N(t)} = t - y] dF(t_{N(t)} | S_N(t) = y) = \mathbb{E}[R_1 | X_1 = t - y] \frac{dF(t - y)}{F(t - y)}.
$$

This established the identity

$$
\mathbb{E}[dR(t)] = \mathbb{E} \left[ R_1 | X_1 = t \right] dF(t) + \int_0^t \left( \mathbb{E}[R_1 | X_1 = t - y] dF(t - y) \right) dm(x).
$$

Now the key renewal theorem gives

$$
\lim_{t \to \infty} \mathbb{E}[dR(t)] = \int_0^t \mathbb{E}[R_1 | X_1 = t] dF(t) dt = \mathbb{E}[R_1] dt.
$$

Thus

$$
\lim_{t \to \infty} \int_t^{t+a} \mathbb{E}[dR(t)] = \int_t^{t+a} \mathbb{E}[R_1] ds = a \frac{\mathbb{E}[R_1]}{\mu}.
$$

Another approach: Note that

$$
\mathbb{E}[\text{reward in } (t, t+a)] = \mathbb{E} \left[ \sum_{n=1}^{N(t+a)} R_n - \sum_{n=1}^{N(t)} R_n \right]
= \mathbb{E} \left[ \sum_{n=1}^{N(t+a) + 1} R_n \right] - \mathbb{E} \left[ \sum_{n=1}^{N(t) + 1} R_n \right] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t)+a+1}]
$$

Now $N(t+a) + 1$ and $N(t) + 1$ are stopping times for the sequence $(X_i, R_i), i = 1, \cdots$. Thus from (generalized) Wald’s equation

$$
\mathbb{E} \left[ \sum_{n=1}^{N(t)+1} R_n \right] = \mathbb{E}[N(t) + 1] \mathbb{E}[R]
$$

and

$$
\mathbb{E} \left[ \sum_{n=1}^{N(t+a)+1} R_n \right] = \mathbb{E}[N(t+a) + 1] \mathbb{E}[R],
$$

where $\mathbb{E}[R]$ is the expected reward in a cycle. Thus

$$
\mathbb{E}[\text{reward in } (t, t+a)] = \mathbb{E}[N(t+a) + 1] \mathbb{E}[R] - \mathbb{E}[N(t) + 1] \mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]
= (m(t+a) - m(t)) \mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}].
$$

Now

$$
\lim_{t \to \infty} (m(t+a) - m(t)) \mathbb{E}[R] = a \mathbb{E}[R] / \mathbb{E}[X].
$$
from Blackwell’s theorem. Now, it suffices to show that \( \lim_{t \to \infty} \mathbb{E}[R_{N(t)+1}] \) exists and is finite. Indeed,

\[
\mathbb{E}[R_{N(t)+1}] = \mathbb{E}[R_{N(t)+1}|S_{N(t)} = 0] \overline{F}(t) + \int_0^t \mathbb{E}[R_{N(t)+1}|S_{N(t)} = s] \overline{F}(t-s)dm(s)
\]

\[
= \mathbb{E}[R|X > t] \overline{F}(t) + \int_0^t \mathbb{E}[R|X > t-s] \overline{F}(t-s)dm(s)
\]

\[
= h(t) + \int_0^t h(t-s)dm(s),
\]

where \( h(t) = \mathbb{E}[R|X > t] \overline{F}(t) \). Then by the Key Renewal theorem, we have

\[
\lim_{t \to \infty} \mathbb{E}[R_{N(t)+1}] = \frac{1}{\mathbb{E}[X]} \int_0^\infty h(s)\,ds.
\]

Here we assumed that \( h(t) \) is directly Riemann integrable.

5. The life of a car is a random variable with distribution \( F \). An individual has a policy of trading in his car either when it fails or reaches the age of \( A \). Let \( R(A) \) denote the resale value of an \( A \)-year-old car. There is no resale value of a failed car. Let \( C_1 \) denote the cost of a new car and suppose that an additional cost \( C_2 \) is incurred whenever the car fails.

(a) Say that a cycle begins each time a new car is purchased. Compute the long-run average cost per unit time.

(b) Say that a cycle begins each time a car in use fails. Compute the long-run average cost per unit time.

Note: In both (a) and (b) you are expected to compute the ratio of the expected cost incurred in a cycle to the expected time of a cycle. The answer should, of course, be the same in both parts.

Solution: (a) Clearly,

\[
\mathbb{E}[\text{cost per cycle}] = C_1 - \overline{F}(A)R(A) + F(A)C_2
\]

and

\[
\mathbb{E}[\text{time of cycle}] = \int_0^A xdF(x) + A(1 - F(A)).
\]

So, treating the cost as the reward, the renewal reward theorem gives

\[
\lim_{t \to \infty} \frac{\mathbb{E}[\text{accumulated cost by } t]}{t} = \frac{\mathbb{E}[\text{cost per cycle}]}{\mathbb{E}[\text{time of cycle}]} = \frac{C_1 - \overline{F}(A)R(A) + F(A)C_2}{\int_0^A xdF(x) + A\overline{F}(A)}
\]

(b) The chance that a car fails is \( F(A) \), so the number, \( N \), of cars bought between failures has the geometric distribution with parameter \( p = F(A) \). We have,

\[
\mathbb{E}[\text{cost per cycle}] = \mathbb{E}[NC_1 - (N-1)R(A) + C_2] = C_1/F(A) + (1 - 1/F(A))R(A) + C_2
\]
and

$$E[\text{time of cycle}] = E[(N-1)A] + E[\text{car life}|\text{car life} < A] = \bar{F}(A)A/F(A) + \int_0^A x\,dF(x)/F(A).$$

Thus,

$$\lim_{t \to \infty} \frac{E[\text{accumulated cost by } t]}{t} = \frac{C_1/F(A) + (1 - 1/F(A))R(A) + C_2}{\bar{F}(A)A/F(A) + \int_0^A x\,dF(x)/F(A)}.$$

Multiplying numerator and denominator by $F(A)$ gives the same expression as in (a).

**Recommended reading:**
Sections 3.4 through 3.7, excluding subsections 3.4.3, 3.6.1, 3.7.1. We will not cover the material in Section 3.8, though you may like to look through it.

**Supplementary exercises:** 3.24, 3.27, 3.35.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.