Homework 6 (Stats 620, Winter 2017)

Due Thursday March 16, in class

1. In a branching process the number of offspring per individual has a Binomial \((2, p)\) distribution. Starting with a single individual, calculate:
   (a) the extinction probability;
   (b) the probability that the population becomes extinct for the first time in the third generation.
   (c) Suppose that, instead of starting with a single individual, the initial population size \(Z_0\) is a random variable that is Poisson distributed with mean \(\lambda\). Show that, in this case, the extinction probability is given, for \(p > 1/2\), by
   \[ \exp\{\lambda(1 - 2p)/p^2\} \]

Instructions: (b) If \(X_n\) is the size of the \(n\)th generation, this question is asking you to find \(P(X_3 = 0, X_2 > 0, X_1 > 0| X_0 = 1)\). This can be done by brute force calculation, or by using probability generating functions.

Solution:
(a) Say \(p_j = \mathbb{P}[X_1 = j| X_0 = 1] = \binom{2}{j}p^j(1-p)^{2-j}\) for \(j = 0, 1, 2\) and 0 for \(j > 2\). Now
   \[ \pi_0 = \mathbb{P}[\text{Population dies out}] = \sum_{j=0}^\infty \pi_j p_j. \]
   Thus
   \[ \pi_0 = (1 - p)^2 + 2\pi_0(1 - p)p + \pi_0^2 p^2. \]
   Solving and choosing the smaller root
   \[ \pi_0 = \begin{cases} 
   1 & p \leq .5 \\
   (1-p)^2 & p > .5 
   \end{cases} \]

(b) Let \(\phi_n(s) = \mathbb{E}[s^{X_n}]\). It was shown in class that \(\phi_n(s) = \phi_1(\phi_{n-1}(s))\). Also its easy to see that \(\phi_1(s) = (sp + 1 - p)^2\). Also \(\mathbb{P}[X_n = 0] = \phi_n(0)\). Thus the probability of extinction in third generation is
   \[ \phi_3(0) - \phi_2(0) = \phi_1(\phi_1(\phi_1(0))) - \phi_1(\phi_1(0)) = 4p^2(1-p)^4 + 6p^3(1-p)^5 + 6p^4(1-p)^6 + 4p^5(1-p)^7 + p^6(1-p)^8 \]

(c) Probability of extinction when \(Z_0 = 1\) and \(p > .5\) is \((1-p)^2\). All families behave independently of each other. Thus when \(Z_0\) has a Poisson distribution with parameter \((\lambda)\), the probability of extinction equals
   \[ \sum_{n=0}^\infty \left[ \frac{(1-p)^2}{p^2} \right]^n P[Z_0 = n], \]
   which is exactly the probability generating function of Poisson r.v. evaluated at \(\frac{(1-p)^2}{p^2}\). Thus probability of extinction equals \(e^{\lambda(1-p)^2/p^2 - 1} \).
2. Consider a time-reversible Markov chain with transition probabilities $P_{ij}$ and limiting probabilities $\pi_i$; and now consider the same chain truncated to the states $0, 1, \ldots, M$. That is, for the truncated chain its transition probabilities $\overline{P}_{ij}$ are

$$\overline{P}_{ij} = \begin{cases} 
    P_{ij} + \sum_{k>M} P_{ik}, & 0 \leq i \leq M, j = i \\
    P_{ij}, & 0 \leq i \neq j \leq M \\
    0, & \text{otherwise.}
\end{cases}$$

Show that the truncated chain is also time reversible and has limiting probabilities given by

$$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i}.$$

**Solution:** Assume that the truncated chain is also irreducible. Simply verify that $$\{\overline{\pi}_i\}_{0 \leq i \leq M}$$ defined by

$$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i},$$

satisfy

$$\overline{P}_{ij}\overline{\pi}_i = \overline{P}_{ji}\overline{\pi}_j, \forall 0 \leq i, j \leq M \quad \text{and} \quad \sum_{i=0}^{M} \overline{\pi}_i = 1..$$

Since the original Markov chain is time– reversible, we have

$$P_{ij}\pi_i = P_{ji}\pi_j, \forall i, j \geq 0.$$

It follows that for any $0 \leq i, j \leq M$, we have

$$\overline{P}_{ij}\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i} \left( P_{ij} + 1_{i=j} \sum_{k>M} P_{ik} \right) = \frac{\pi_j}{\sum_{i=0}^{M} \pi_i} \left( P_{ji} + 1_{i=j} \sum_{k>M} P_{jk} \right) = \overline{P}_{ji}\overline{\pi}_j.$$

3. For an ergodic semi-Markov process:

   (a) Compute the rate at which the process makes a transition from $i$ into $j$.

   (b) Show that $\sum_i P_{ij}/\mu_{ii} = 1/\mu_{jj}$.

   (c) Show that the proportion of time that the process is in state $i$ and headed for state $j$ is $P_{ij}\eta_{ij}/\mu_{ii}$ where $\eta_{ij} = \int_0^\infty F_{ij}(t) \, dt$.

   (d) Show that the proportion of time that the state is $i$ and will next be $j$ within a time $x$ is

$$\frac{P_{ij}\eta_{ij}}{\mu_{ii}} F_{ij}^e(x),$$

where $F_{ij}^e$ is the equilibrium distribution of $F_{ij}$

**Hint:** all parts of this question can be done by defining appropriate renewal-reward processes.

For (d), we use the definition $F_{ij}^e(x) = \int_0^x F_{ij}(y) \, dy / \int_0^\infty F_{ij}(y)$ (see Ross, p131). This is the delay required to make a delayed renewal process with renewal distribution $F_{ij}$ stationary. It arises here since it is also the limiting distribution of the residual life process for a non-lattice renewal process.
Solution: (a) Define a (delayed) renewal reward process: a renewal occurs when state $i$ is entered from other states and the reward of each $n$–th cycle $R_n$ equals 1 if in the $n$–th cycle, the state after $i$ is $j$ and 0 otherwise. Let $R_{ij}(t)$ be the total number of transitions from $i$ to $j$ by time $t$. We have

$$\sum_{n=0}^{N(t)} R_n \leq R_{ij}(t) \leq \sum_{n=0}^{N(t)+1} R_n \leq R_{ij}(t) + 1.$$ 

Thus the rate at which the process makes a transition from $i$ to $j$ equals

$$\lim_{t \to \infty} \frac{R_{ij}(t)}{t} = \frac{E[R]}{E[X]} = \frac{P_{ij}}{\mu_{ii}}.$$ 

(b) Let $R_j(t)$ be the number of visits to state $j$ by time $t$. Thus

$$\sum_i R_{ij}(t) = R_j(t)$$

$$\sum_i \lim_{t \to \infty} \frac{R_{ij}(t)}{t} = \lim_{t \to \infty} \frac{R_j(t)}{t}$$

$$\sum_i \frac{P_{ij}}{t} = \frac{1}{\mu_{jj}}.$$ 

(c) Define cycle as in part (a) and the reward in a cycle to be 0 if the transition from $i$ is not into $j$ and $T_{ij}$ the time taken for transition if the transition from $i$ is into $j$. Thus

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]} = \frac{P_{ij}E[T_{ij}]}{\mu_{ii}} = \frac{P_{ij}\eta_{ij}}{\mu_{ii}}.$$ 

(d) Define cycle as in last part and the reward in a cycle as 0 if the transition from $i$ is not into $j$ and $\min(x,T_{ij})$ if the transition from $i$ is into $j$. Thus

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]} = \frac{P_{ij}E[\min(x,T_{ij})]}{\mu_{ii}} = \frac{P_{ij}\eta_{ij}}{\mu_{ii}} F_{ij}^c(x).$$

4. A taxi alternates between three locations. When it reaches location 1 it is equally likely to go next to either 2 or 3. When it reaches 2 it will next go to 1 with probability 1/3 and to 3 with probability 2/3. From 3 it always goes to 1. The mean times between location $i$ and $j$ are $t_{12} = 20, t_{13} = 30$ and $t_{23} = 30$ (with $t_{ij} = t_{ji}$).

(a) What is the (limiting) probability that the taxi’s most recent stop was at location $i$, $i = 1, 2, 3$?

(b) What is the (limiting) probability that the taxi is heading for location 2?

(c) What fraction of time is the taxi traveling from 2 to 3? Note: Upon arrival at a location the taxi immediately departs.

Solution: First we write the transition matrix:

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1/3 & 0 & 2/3 \\ 1 & 0 & 0 \end{bmatrix}.$$
Now the stationary probabilities can be found by solving $\pi = \pi P$. We have

$$\pi_1 = \frac{6}{14}, \pi_2 = \frac{3}{14}, \text{ and } \pi_3 = \frac{5}{14}.$$  

Since $\mu_i = \sum P_{ij}\mu_j$, we have $\mu_1 = 25$, $\mu_2 = \frac{80}{3}$, and $\mu_3 = 30$.

(a) By formula

$$P_i = \frac{\pi_i\mu_i}{\sum_j \pi_j\mu_j},$$

we have

$$P_1 = \frac{15}{38}, P_2 = \frac{8}{38}, \text{ and } P_3 = \frac{15}{38}.$$  

They are the correspondingly required limiting probabilities.

(b) Use part (c) of 4.48. Since the taxi can only go to location 2 from location 1, the limiting probability that taxi is headed for location 2 equals

$$P_{12}\eta_{12}(\mu_1 P_1)^{-1} = \frac{3}{19}.$$  

(c) Same argument as in part (b) implies that the proportion of the time that the taxi is traveling from location 2 to location 3 equals

$$P_{23}\eta_{23}(\mu_2 P_2)^{-1} = \frac{3}{19}.$$  

**Recommended reading:**
Sections 4.5, 4.7, 4.8, 5.1, 5.2. You may skip Section 4.6, which will not be covered in this course.

**Supplementary exercise:** 4.40
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.