Solutions to Homework 12

7.13 The chance a person’s age matches the card number is 1/1000. By (3.1) the expectation is 1.

7.19 a) The number of insects caught has a geometric distribution with parameter \( P \) and mean \( 1/P \). The number before the first type 1 catch is one smaller with mean \( (1 - P)/P \). b) The chance type \( j \) occurs before type 1 is \( P_j/(P_1 + P_j) \), so by (3.1) the mean is \( \sum_{j=2}^{r} P_j/(P_1 + P_j) \).

7.21 a) The probability there are exactly 3 people with birthdays on day \( i \) is \( \binom{100}{3} \left( \frac{1}{365} \right)^3 \left( \frac{364}{365} \right)^{97} = 0.002548 \). Multiplying this by 365, by (3.1) the expectation is 0.93. b) The chance of a birthday on day \( i \) is \( 1 - \left( \frac{364}{365} \right)^{100} = 0.24 \), so by (3.1) the expected number of distinct birthdays is 87.58.

7.25 The event \( N > n \) occurs if and only if \( X_1 \geq X_2 \geq \cdots \geq X_n \). By symmetry, possible orderings of \( X_1, \ldots, X_n \) are equally likely, and so

\[
P(N > n) = P(X_1 \geq X_2 \geq \cdots \geq X_n) = \frac{1}{n!}.
\]

Using this,

\[
P(N = n) = P(N > n - 1) - P(N > n) = \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n-1}{n!}.
\]

Thus

\[
EN = \sum_{n=2}^{\infty} \frac{n(n-1)}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = \sum_{j=0}^{\infty} \frac{1}{j!} = e.
\]

7.26 a) If \( Y = \max\{X_1, \ldots, X_n\} \), then for \( y \in (0,1) \),

\[
P(Y \leq y) = P(X_1 \leq y, \ldots, X_n \leq y) = P(X_1 \leq y) \times \cdots \times P(X_n \leq y) = y^n.
\]

So \( f_Y(y) = F_Y'(y) = ny^{n-1}, \ y \in (0,1) \) and

\[
EY = \int y f_Y(y) dy = \int_0^1 y ny^{n-1} dy = \frac{n}{n+1}.
\]

b) If \( W = \min\{X_1, \ldots, X_n\} \), then for \( w \in (0,1) \),

\[
P(W \leq w) = 1 - P(W > w) = 1 - P(X_1 > w, \ldots, X_n > w)
\]

\[
= 1 - P(X_1 > w) \times \cdots \times P(X_n > w) = 1 - (1 - w)^n.
\]

So \( f_W(w) = F_W'(w) = n(1-w)^{n-1}, \ w \in (0,1) \) and

\[
EW = \int w f_W(w) dw = \int_0^1 w n(1-w)^{n-1} dw = \frac{1}{n+1}.
\]

7.30 Since \( EX = EY \), \( E(X - Y) = 0 \). Using this and the independence,

\[
E(X - Y)^2 = \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(-Y) = \text{Var}(X) + \text{Var}(Y) = 2\sigma^2.
\]
7.31 If $X_i$ is the $i$th roll, then $E X_i = 3.5$,
\[ E X_i^2 = \frac{1}{6}(1 + 4 + \cdots + 36) = 91/6, \]
and $\text{Var}(X_i) = (91/6) - (3.5)^2 = 35/12$. By the independence,
\[ \text{Var}(X_1 + \cdots + X_{10}) = \text{Var}(X_1) + \cdots + \text{Var}(X_{10}) = 350/12. \]

7.37 Using Proposition 4.2,
\[ \text{Cov}(R_1 + R_2, R_1 - R_2) = \text{Cov}(R_1, R_1) - \text{Cov}(R_1, R_2) + \text{Cov}(R_2, R_1) - \text{Cov}(R_2, R_2) = \text{Var}(R_1) - 0 + 0 - \text{Var}(R_2) = 0. \]

7.38 Integrating against $f$,
\[
\begin{align*}
EX &= \iint x f(x, y) \, dy \, dx = \int_0^\infty \int_0^x 2e^{-2x} \, dy \, dx = \int_0^\infty 2xe^{-2x} \, dx = 1/2, \\
EY &= \iint y f(x, y) \, dy \, dx = \int_0^\infty \int_0^x 2(y/x)e^{-2x} \, dy \, dx = \int_0^\infty xe^{-2x} \, dx = 1/4,
\end{align*}
\]
and
\[
E(XY) = \iint xy f(x, y) \, dy \, dx = \int_0^\infty \int_0^x 2ye^{-2x} \, dy \, dx = \int_0^\infty x^2e^{-2x} \, dx = 1/4.
\]
So $\text{Cov}(X, Y) = E(XY) - (EX)(EY) = 1/8$.

7.40 Integrating against $f$,
\[
\begin{align*}
EX &= \iint x f(x, y) \, dy \, dx = \int_0^\infty \int_0^\infty (x/y)e^{-(y+x)/y} \, dx \, dy = \int_0^\infty y e^{-y} \, dy = 1, \\
EY &= \iint y f(x, y) \, dy \, dx = \int_0^\infty \int_0^\infty e^{-(y+x)/y} \, dx \, dy = \int_0^\infty y e^{-y} \, dy = 1, \\
E(XY) &= \iint xy f(x, y) \, dy \, dx = \int_0^\infty \int_0^\infty e^{-(y+x)/y} \, dx \, dy = \int_0^\infty y^2 e^{-y} \, dy = 2.
\end{align*}
\]
So $\text{Cov}(X, Y) = E(XY) - (EX)(EY) = 1$.

7.42 a) Let $X$ be the number of pairs with a man and a woman, and let $A_i$ be the event that pair $i$ consists of a man and a woman. Since pair $i$ is equally likely to be any of the $\binom{20}{2} = 190$ pairs, 100 of which have a man and a woman, $P(A_i) = 10/19$, and by (3.1), $EX = 100/19 = 5.263$. Since there are $\binom{20}{2}\binom{18}{2} = 29070$ equally likely possibilities for pairs $i$ and $j$, and for $10^2 9^2 = 8100$ of these both pairs consist of a man and a woman, $P(A_iA_j) = 8100/29070 = 0.2786$. So by (3.3), $EX^2 - EX = 25.077$. Hence
\[ \text{Var}(X) = 25.077 + 5.263 - (5.263)^2 = 2.641. \]

b) Let $Y$ be the number of married couples that are paired together, and let $B_i$ be the event that the two people for pair $i$ are married. Then $P(B_i) = 10/190$, and by (3.1), $EY = 0.526$. Since $P(B_iB_j) = 90/29070 = 0.003096$, by (3.3), $EY^2 - EY = 0.2786$. Hence
\[ \text{Var}(Y) = 0.2786 + 0.526 - (0.526)^2 = 0.528. \]
8.1 By Chebyshev’s inequality,

\[ P(0 < X < 40) = 1 - P(|X - 20| \geq 20) \geq 1 - \frac{20}{20^2} = 0.95. \]

8.4 a) Let \( S = \sum_{i=1}^{20} X_i \). Then \( ES = 20 \), and by Markov’s inequality, \( P(X > 15) < 20/15 = 4/3 \).

(Not very useful, since the probability is at most one.) b) Since \( \text{Var}(S) = 20 \), by the CLT, \( S \) is approximately \( N(20, 20) \) and (with a continuity correction)

\[ P(S > 15) = P(S > 15.5) = P\left( \frac{S - 20}{\sqrt{20}} > \frac{15.5 - 20}{\sqrt{20}} = -1.01 \right) \approx 84.38\%. \]

8.6 Let \( S \) be the sum of the first 79 rolls. Then the event that at least 80 rolls are necessary is the same as the event that \( S \leq 300 \). If \( X \) is the number of pips showing when a die is rolled once, then

\[ EX = \frac{1}{6} \times 1 + \cdots + \frac{1}{6} \times 6 = 7/2, \quad EX^2 = \frac{1}{6} \times 1^2 + \cdots + \frac{1}{6} \times 6^2 = \frac{91}{6}, \quad \text{Var}(X) = \frac{91}{6} - \left( \frac{7}{2} \right)^2 = \frac{35}{12}. \]

By the CLT, \( S \) is approximately normal with mean \( \mu_S = 79 \times (7/2) = 276.5 \) and variance \( \sigma_S^2 = 79 \times (35/12) = 230.4 \). So

\[ P(S \leq 300) = P(S \leq 300.5) = P\left( \frac{S - \mu_S}{\sigma_S} \leq \frac{300.5 - \mu_S}{\sigma_S} = 1.58 \right) \approx \Phi(1.58) = 0.9429. \]

8.9 Let \( Y = (X - n)/\sqrt{n} \). Since the sum of \( n \) independent exponential variables with \( \lambda = 1 \) has the gamma distribution with parameters \( \alpha = n \) and \( \beta = 1 \), by the central limit theorem the distribution function for \( Y \) is approximately \( \Phi \). So

\[ P\left\{ \left| \frac{X}{n} - 1 \right| > 0.01 \right\} = P(|Y| > 0.01\sqrt{n}) = P(Y > 0.01\sqrt{n}) + P(Y < -0.01\sqrt{n}) \approx 1 - \Phi(0.01\sqrt{n}) + \Phi(-0.01\sqrt{n}) = 2[1 - \Phi(0.01\sqrt{n})]. \]

This will be 0.01 if \( \Phi(0.01\sqrt{n}) = 0.995 \). From the table, this will be the case if \( 0.01\sqrt{n} = 1.96 \), or if \( n = 196^2 = 38,416 \).

8.15 By the central limit theorem, total claims \( S \) should be approximately normal with mean \( \mu_S = $2,400,000 \) and variance \( \sigma_S^2 = 10000 \times 800^2 = 6,400,000,000 \). So

\[ P(S > 2,700,000) = 1 - P(S \leq 2,700,000) \]

\[ = 1 - P\left( \frac{S - \mu_S}{\sigma_S} \leq \frac{2,700,000 - \mu_S}{\sigma_S} = 3.75 \right) \approx 1 - \Phi(3.75) = 0.00009. \]

(Table in the book does not go this far.)