Announcement: For purposes of the homework, you can cite any results in the handouts or the text-book or any others proved in class, without proof. The homework carries a total of 50 points, but contributes 5 points towards your total grade.

- 1. Prove that for three not necessarily disjoint events $A$, $B$ and $C$,

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C). \]

**Hint:** You can write $A \cup B \cup C$ as $(A \cup B) \cup C$ and use the formula for the union of two events (on page 2 of the first handout) and proceed from there. (5 points)


- 3. (a) Show that if events $A_1, A_2, \ldots, A_n$ are mutually independent, then so are $A_1, A_2, \ldots, A_{n-1}, A_n^c$. (Hint: Use the definition of mutual independence)

(b) Use this result repeatedly to show that if $B_1, B_2, \ldots B_n$ are independent events then so are $C_1, C_2, \ldots, C_n$ where each $C_i$ is either $B_i$ or $B_i^c$. (Hint: Observe that it suffices to prove that if $B_1, B_2, \ldots B_n$ are independent, then so are $B_1, B_2, \ldots, B_m, B_m^c, B_m^c, \ldots, B_n^c$. (Why ?)) (3 + 2 = 5 points)

- 4. We call $X$ a geometric random variable if $X$ takes values \{1, 2, 3, \ldots\} and

\[ P(X = m) = pq^{m-1}, \text{ where } 0 < p, q < 1 \text{ and also } p + q = 1. \]

Refer to the handout for a random experiment that produces a geometric random variable.

(a) Prove that for any two positive integers $m, n$, it is the case that,

\[ P(X > m + n \mid X > m) = P(X > n). \]

This is the **memoryless property** which is discussed a bit in the Probability Refresher notes. To show this, first prove that the memoryless property is equivalent to the assertion that

\[ P(X > m + n) = P(X > m) P(X > n). \]
Next, show that for the geometric distribution, for any positive integer \( l \),

\[ P(X > l) = q^l, \]

and proceed.

(b) We will prove the converse of (a). We will show that if \( X \) is a discrete random variable taking values \( \{1, 2, 3, \ldots\} \) with probabilities \( \{p_1, p_2, p_3, \ldots\} \) and satisfies the memoryless property, then \( X \) must follow a geometric distribution.

Follow these steps to establish the fact that \( X \) is geometric. Using the fact that \( X \) has the memoryless property, show that

\[ P(X > m) = (P(X > 1))^m, \]

for any \( m \geq 2 \). As a first step towards proving this show that

\[ P(X > 2) = (P(X > 1))^2. \]

Define \( p = P(X = 1) \) and \( q = P(X > 1) \). You now have,

\[ P(X > m) = q^m, \]

for any \( m \geq 2 \). Use this to show that for any \( m \geq 2 \),

\[ P(X = m) = pq^{m-1}. \]

**Hint:** Note that the event \( \{X > m - 1\} \) is the disjoint union of the events \( \{X > m\} \) and \( \{X = m\} \).

But for \( m = 1 \),

\[ P(X = m) = P(X = 1) = p = pq^{m-1}, \]

trivially and the proof is complete. (5 + 5 = 10 points)

• 5. If \( X \) is random variable with distribution function \( F \), with continuous non-vanishing density \( f \), obtain the density function of the random variable \( Y = X^2 \), from first principles; i.e. **without** using the extended change of variable theorem on Page 14 of the first handout.

**Hint:** Express the probability of the event \( (X^2 \leq y) \) in terms of the distribution function \( F \) of \( X \) and proceed from there. (5 points)
6. (i) Let $X$ be a continuous random variable with distribution function $F$. Let $f(x) = F'(x)$ denote the density function of $X$. Assume that this is continuous and never vanishes. Use the change of variable theorem to show that $Y = F(X)$ has the uniform distribution on $(0, 1)$.

(ii) Let $T$ be an exponential random variable with parameter $\lambda$ and let $W$ be a random variable independent of $T$ which assumes the value 1 with probability $1/2$ and the value $-1$ with probability $1/2$. Show that the density of $X = WT$ is,

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

using first principles. This distribution is called the double exponential distribution.

Hint: It would help to split up the event \{\(X \leq x\)\} as the union of \{\(X \leq x, W = 1\)\} and \{\(X \leq x, W = -1\)\}. (5 + 5 = 10 points)

7. Consider a population of $N$ voters who will either vote for Democrats or Republicans (i.e. no one abstains from voting, or cancels their vote). The goal is to estimate the proportion $p$ of Democrats in the population.

**Sampling without replacement:** A sample of size $n$ is collected at random without replacement from this population. Let $X_i$ denote the affiliation (1 if Democrat, 0 if Republican) of the $i$'th individual in the sample.

(i) Write down the p.m.f of $X_1$ and the joint p.m.f. of $(X_1, X_2)$. Compute the p.m.f. of $X_2$ and show that it is identical to the p.m.f. of $X_1$. Also show that $X_1$ and $X_2$ are not independent.

In general, $X_1, X_2, \ldots, X_n$ all have identical marginal distributions but are dependent. Also, the joint p.m.f. of $(X_i, X_j)$ is the same for all pairs $(i, j)$.

(ii) Consider the special case when the sample size $n$ is equal to the population size $N$, so that your random sample is $(X_1, X_2, \ldots, X_N)$. Compute $E(S)$ and $\text{Var}(S)$.

(iii) Let $\hat{p} = n^{-1}(X_1 + X_2 + \ldots + X_n)$ be the natural estimate of $p$. Compute $E(\hat{p})$ and $\text{Var}(\hat{p})$. (10 points)